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Keywords: Two-agents, Approval Voting, Efficiency, Partial Honesty

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Bargaining through Approval*

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Abstract

The paper considers two-person bargaining under Approval Voting. It first proves the existence of pure strategy equilibria. Then it shows that this bargaining method ensures that both players obtain at least their average and median utility level in equilibrium. Finally it proves that, provided that the players are partially honest, the mechanism triggers sincerity and ensures that no alternative Pareto dominates the outcome of the game.

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1 Introduction

We study a very simple and intuitive mechanism for two players which is shown to exhibit appealing properties. This mechanism, the Approval Mechanism, is a one-step procedure in which each player announces a subset of the alternatives as their "approved" ones, the most approved alternatives are declared winners, and ties are broken with a uniform lottery. The Approval Mechanism is simply the Approval Voting rule with two players.

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Voting rules are usually conceived for many players,[1] but the two-person case also has practical interest for resolving two-person disputes. Potential applications may also include jury selection in the US or agenda-setting situations in the political arena when there are only two dominant parties. To our knowledge, ours is the first work to explore the properties of Approval Voting as a two-player bargaining device.

Given the tie-breaking rule, we thus consider the following setting:

(i) there is a finite set of pure alternatives,

(ii) two individuals have von Neumann–Morgenstern preferences over pure alternatives,

(iii) the game outcomes are the uniform lotteries[2] over subsets of alternatives.

This framework is hence different from the usual ones in bargaining or implementation theory in which it is usually assumed that the game outcome can be either any of the alternatives or any possible lottery over the alternatives[3].

We use pure strategy equilibria as a predictive device. Despite the two players moving simultaneously, a pure strategy equilibrium exists for any preference profile. The proof is constructive and proceeds as follows: it first builds a sequence of iterated best responses and then proves that this sequence leads to an equilibrium. Moreover, the equilibrium obtained by this construction is sincere. We prove that there can be two types of equilibria: consensual and non-consensual ones. In a consensual equilibrium, both agents announce a single common alternative, which is thus outcome. In a non-consensual one, the sets of announced alternatives are disjoint and there is thus a tie among several alternatives.

[1] For recent theoretical work on Approval Voting in elections with a large number of voters, see Myerson [2002], Laslier [2009], Núñez [2010, 2013], Goertz and Maniquet [2011], Bouton and Castanheira [2012], Courtin and Núñez [2013] and Ahn and Oliveros [2011] among others.

[2] Our results still hold if we relax these assumptions according to which lotteries are uniform or the agents have vN-M utilities. Section 8.2 proves that the current results are valid if one suitably weakens the previously described assumptions.

[3] These literatures are vast, and we do not attempt here to give a full review. Several works have studied bargaining over a finite set of alternatives. Most of them take an ordinal approach and leave preference intensities out of the setting. Among those, sequential procedures (in particular the "fallback" bargaining method) have attracted a considerable interest (Sprumont [1993], Hurwicz and Sertel [1999], Brams and Kilgour [2001], Anbarci [2006],Kıbrıs and Sertel [2007] and De Clippel and Eliaz [2012]). A related literature considers problems with a finite number of alternatives and focuses on cardinal rules, as we do. The Nash and the Kalai-Smorodinsky solutions have been characterized in this setting (Mariotti [1998]; Nagahisa and Tanaka [2002]).

We then prove that this rule satisfies three normative properties: *Random Lower Bound* (*RLB*), *Sincerity* (S) and *Pure Pareto Efficiency* (PPE).

The first property, *RLB*, is the natural adaptation of the classical axiom of Equal-Division Lower Bound in fair allocation settings (see for instance Kolm [1973], Pazner [1977] and Thomson [2010] for a review of the literature). It simply states that a mechanism should always assign to a player at least the expected level of utility she would obtain if the selected alternative was chosen at random, uniformly among all available alternatives. In other words, in any equilibrium, the expected utility for each player is at least equal to her mean utility, which would be achieved using the uniform lottery over the whole set of alternatives. Random Lower Bound hence ensures a minimal level of utility to both players, from an ex-ante point of view.

It can be argued that this property is quite mild since, intuitively, it seems that one could design mechanisms that ensure strictly more utility than the random lower bound to both players. However, we prove that this intuition is wrong since, simply, such an outcome need not exist for all utility profiles. In other words, the random lower bound is the highest level of minimal utility for both players one can ensure.

Moreover, in our setting, in which the outcomes coincide with the uniform lotteries, the mean expected utility coincides with the median one. This precludes the discussion of whether it is fairer to require that a mechanism deliver the mean vs. the median expected utility.[4]

As to *Sincerity*, we use the classical definition under Approval Voting: a ballot is sincere when a player approves and alternative, she also approves all the alternatives she ranks higher. We first recall that a strategic player may vote non-sincerely in a Nash equilibrium. Yet, this sincerity violation is shown to be quite mild in the following sense of "partial honesty". The idea of *partial honesty* is simple (see Matsushima [2008], Dutta and Sen [2012], Kartik and Tercieux [2012] and **?** for applications in Nash implementation). A partially honest player is one who votes sincerely when voting sincerely maximizes her utility. Suppose, for instance, that a player hesitates between voting for her preferred alternative or approving of just

[4] De Clippel et al. [2013] design sequential mechanisms that satisfy a related property (the "minimal satisfaction" condition): these mechanisms deliver at least the median pure utility level to both players (see also **??** for recent work on sequential procedures based on lists). Similarly, Conley and Wilkie [2012] define the ordinal egalitarian solution with finite choice sets. This solution suggests the middle ranked alternative (or a lottery over the two middle ranked alternatives) of the Pareto set as an outcome.

her first and third preferred ones (a non-sincere ballot). Suppose that both ballots deliver the same outcome, so that a classical strategic player would be indifferent. In this case, a partially honest player will not cast the non-sincere ballot. Such a player has a very limited preference for honesty: she acts honestly only if this is not detrimental to her. We will show that, in our setting, the behavioral assumption of partial honesty removes insincerity in equilibrium: if players are partially honest, they are sincere in every equilibrium.

As far as Pareto Efficiency is concerned, the first remark is that, with just three alternatives, this mechanism is Lottery Pareto efficient in the sense that no lottery Pareto dominates, in the usual sense, an equilibrium of the game. Yet, this result does not extend to more than three alternatives. However, the Approval Mechanism satisfies the weaker notion of *Pure Pareto Efficiency*. We will prove that no equilibrium of the game can be Pareto dominated by a (pure) alternative as long as the players are partially honest. Note that the previous property singles out the Approval mechanism. We will prove that there does not exist a social choice correspondence that satisfies Random Lower-Bound and Pure Pareto Efficiency while being fully implementable in pure strategies (independently of whether the agents are or not partially honest). Indeed, the Approval mechanism partially implements the set of uniform lotteries that jointly satisfy the previously mentioned conditions.

We finally consider two variations of the mechanism. We first focus on whether our results extend to mixed strategies. Whereas *RLB* still holds, the sincerity of players' best responses is not anymore valid. This is shown by an example which proves that mixed strategies might trigger counterintuitive probability distributions over the different pivotal outcomes.[5] This hints at the following conclusion: equilibria in pure strategies are the most adequate framework to study the current mechanism.

The second variation concerns the uniform tie-breaking rule. In this extension, we relax this assumption and allow for alternative rules. More specifically, we give two axioms that determine how the players derive their preferences over the sets of alternatives from their preferences over single alternatives. Under these axioms, we prove that the mechanism still admits a pure strategy equilibrium. Moreover, the existence of a sincere best response is still ensured. This variation show that our results are not tight to the assumption of uniform tie-breaking nor to the assumption

[5] The possibility of such paradoxical but rational behavior under Approval Voting already appeared in the game-theoretical literature in De Sinopoli et al. [2006].

of VNM evaluation of outcomes.

The rest of this work is structured as follows. After discussing an example on the potential applications of the mechanism in Section 2, Section 3 presents the setting, and the proof of existence of pure strategy equilibria is included in Section 4. Section 5 shows that the Approval Mechanism satisfies the *RLB* property, and Section 6 deals with the concept of partial honesty. Section 7 presents the results related to Pareto efficiency and Section 8 describes the previously described extensions. Finally, Section 9 offers some concluding comments.

2 An example

The objective of the following example is to show the usefulness of the Approval mechanism: it correctly aggregates the players' intensities of preferences while satisfying a minimal degree of utility to both of them. This example is very much related to the framework of **?** which study compromising under incomplete information[6].

There are two parties, say *B*lue and *R*ed, who have to choose which topic they would include on today's agenda. There are three possible topics : education, health and taxation, respectively denoted e, h and t. Only one topic can be included in the agenda. The parties have perfectly opposed preferences with B preferring e to h and h to t and R having the inverse preferences. To represent these preferences, we endow each party with a utility vector that represents his cardinal utilities over each of the different items, as follows:

$$u_B = (100, x, 0) \text{ and } u_R = (0, y, 100),$$

with $0 < x, y < 100$.

Given this conflict situation, we are concerned with two basic questions:

(a) Is there an outcome more desirable for both players than a random draw?

(b) Can we achieve this outcome in the equilibrium of a mechanism?

[6]More specifically, they study the case in which two agents have to choose one among the three available alternatives. They assume that the agents' preferences are perfectly opposed and find an impossibility result in the sense that mechanisms which "truthfully elicit utilities and implement efficient decisions do not exist" (efficiency is defined in an utilitarian manner). See also **?** for a related work.

The Approval procedure

We answer both questions in the affirmative.

As far as the question (a) is concerned, note that both players' preferences seem irreconcilable. Thus, the uniform draw (picking one of the options at random) seems to be particularly interesting. This outcome implies that each party gets, in expected utility, her average outcome. Therefore, the random draw ensures each player a minimal amount of utility and seems to be a good benchmark.[7]

Yet, depending on the players' intensities of preferences, *both players* might strictly prefer some other lottery. For instance, it might be the case that both players prefer their middle rank option h to the random draw eht (when $x, y > 50$). In this case, using the random draw seems less well-grounded. As depicted by the next table, for each specification of the preference intensities (x and y), there is a unique lottery that delivers each player at least his average expected utility while not being Pareto dominated.

Preference Intensities	*Outcome*
$x, y > 50$	h
$x > 50$ and $y < 50$	eht
$x < 50$ and $y > 50$	eht
$x, y < 50$	et

Regarding question (b), the answer is also affirmative: for all parameter values, the Approval Mechanism has, in this example, a unique equilibrium, whose outcome coincides with the lottery described by the table.

To get the intuition, consider the case in which $x, y > 50$. Let us recall that each player selects a subset of the alternatives. Since both players prefer h to the lottery eht, it follows that independently of the undominated strategy of the opponent, they always select their most preferred alternative and their middle-ranked one (i.e. h). This proves that in equilibrium, the unique outcome is a consensus over h. A similar claim applies to the rest of specifications of preference intensities, proving the claim.

In general, that is for any number of alternatives, we prove that the mechanism satisfies the following properties which underline its interest for both theoretical and practical work. The outcome is at least as good, for both players, as a random draw, and is not Pareto dominated by a pure alternative. Moreover, under the mild

[7] See Eliaz and Rubinstein [2013] for experimental work on the fairness of random procedures.

assumption of partial honesty, a player is sincere in equilibrium in the usual sense: if he includes an alternative on his ballot, he also includes all the options he prefers to it.

Alternative Procedures

Up to now, we have not justified our focus on the Approval Mechanism. Can other procedures lead to the same equilibria outcomes? As we now discuss, this seems hard to achieve at least with simple procedures.

First, observe that no deterministic mechanism can achieve the desiderata of being Pareto efficient and ensuring a minimal amount of utility to both voters, since in the example, the unique possible outcome may require in some cases to use a lottery. Note that the same limitation also applies to the Rubinstein's alternative vetoes-offers bargaining mechanism. This mechanism implies in our setting that players alternatively offer one pure alternative and then decide whether to accept or reject his opponent's offer. This procedure is either infinitely repeated or repeated a finite number of times in which some exogenous payoff is allowed to both players. Therefore, the outcome can be either a pure alternative or some exogenous disagreement point. Thus, the sort of inefficiency in which we focus in this work seems unavoidable.

Second, observe that our idea consists of using a voting device in this sort of environment. What prevents us from letting the players using other voting rules?

Take the most studied voting rules: the positional or scoring rules. Under such rules in a three-alternative context, a player chooses a vector $(1, s, 0)$, $s \in [0, 1]$, and assigns 1 point to one alternative, s points to some other alternative and 0 to the remaining one. The scores are summed, and the alternative with the most votes is elected. For instance, Plurality Voting corresponds to $s = 0$, the Borda rule to $s = 1/2$, and Negative Voting to $s = 1$.

In the agenda-setting situation, we prove that for some specification of the players' intensities, no equilibrium under these rules satisfies our desiderata. In other words, no scoring rule can achieve the same outcome as the one obtained through the Approval Mechanism.[8]

Take first the Plurality rule with vector $(1, 0, 0)$. Each player is always better-off by voting for his most preferred alternative (he is pivotal independently of the

[8] The equilibria we find under these rules follow closely the equilibrium behavior described by Myerson [2002] in Poisson games.

other player's strategy); the unique equilibrium outcome is the lottery et, which is inefficient if for instance $x, y > 50$. A similar argument holds for any $s \in [0, 1/2)$, proving that none of these rules can be optimal in the previously described sense.

Take now the Negative Voting method with vector $(1, 1, 0)$. Simple computations prove that the unique equilibrium is that both voters vote for their two better alternatives, which leads to the victory of h. This outcome is inefficient unless $x, y > 50$. A similar argument remains true as long as $s \in (1/2, 1]$, proving that every equilibrium leads to the victory of h implying that these rules are inefficient.

Finally, we need to address the case under the Borda rule in which $s = 1/2$. Note first that there is no equilibrium with this rule in which the outcome equals et. Indeed, if the outcome is et, the players have voted $(1, 0, 1/2)$ and $(1/2, 0, 1)$. Hence, each player can attain the victory of his preferred alternative by deviating to his sincere strategy. However, as previous discussed, if $x, y < 50$, et is the efficient outcome, proving that this rule is also inefficient.

3 The Game

There are two players $i = 1, 2$ and a finite set $X = \{x, y, ...,\}$ of at least two alternatives. Both players vote simultaneously. A player can approve as many alternatives as she wishes by choosing a vector B_i from the set of pure strategy vectors $\mathcal{B}_i = 2^X$. For instance, with three alternatives, we have:

$$B_i \in \Big\{(1,0,0),(0,1,0),(0,0,1),(1,1,0),(1,0,1),(0,1,1),(0,0,0),(1,1,1)\Big\}.$$

Note that each ballot can also be represented as a subset of the set of alternatives, so that we often use the notation $B_i \subseteq X$.

A strategy profile $B = (B_1, B_2)$ belongs to $\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2$. Let $s_x(B)$ be the number of votes (the score) for alternative x if the strategy profile is B, and let $s_x(B_{-i})$ be the score for alternative x when the vote of i is excluded. The vector $s(B) = (s_x(B))_{x \in X}$ stands for the total score vector. Note that for each $x \in X$, $s_x(B) \in \{0, 1, 2\}$ since there are just two players. The winning set of alternatives, that is, the outcome corresponding to B, is denoted by $W(B)$ and consists of those alternatives that get the maximum total score:

$$W(B) = \{x \in X \mid s_x(B) = \max_{y \in X} s_y(B)\}.$$

Each player $i \in N$ is endowed with a strict ordering over the set of alternatives X. We assume that preferences can be represented by a von Neumann–Morgenstern utility function $u_i : X \to \mathbb{R}$ over lotteries on the set of alternatives.[9] Each u_i belongs to $\mathcal{U}_i$, the set of utilities for a player. Given that ties are broken by a fair lottery, the expected utility of a player i is a function of the strategy profile B given by:

$$U_i(B) = \frac{1}{\#W(B)} \sum_{x \in W(B)} u_i(x).$$

Slightly abusing notation, we write $U_i(W)$ to denote the expected utility of a player i corresponding to the winning set $W \subseteq X$. Letting $u = (u_1, u_2) \in \mathcal{U}_1 \times \mathcal{U}_2$, the strategic form voting game Γ is then defined by $\Gamma = (u, \mathcal{B})$.

We impose the following condition to ensure that each player has a well-defined strict preference order over the different winning sets: $\forall i, B, B'$ with

$$W(B) \neq W(B') \Longrightarrow U_i(W(B)) \neq U_i(W(B')). \tag{1}$$

This condition implies that no player is indifferent between any pair of alternatives, and moreover no player is indifferent between any pair of winning sets. The condition is generically satisfied with respect to the values of the utilities. For instance it is satisfied with probability 1 if one picks (u_1, u_2) from a probability distribution continuous with respect to the Lebesgue measure in $\mathbb{R}^{2\#X}$. We thus refer to this assumption as a *genericity* assumption.

A *unanimous* society[10] u is a utility vector with for some $y \in X$, $u_i(y) > u_i(x)$ $\forall x \in X \setminus \{y\}$, and a *non-unanimous* society is a utility vector which is not unanimous.

We mainly focus on pure strategy equilibria. An equilibrium is, as usual, a strategy profile $B = (B_1, B_2)$ such that each player is playing a best response to the other player's strategy.

[9] Since the game outcomes are the uniform lotteries over the elements of X, one can weaken the assumption of von Neumann–Morgenstern preferences without altering the results (see **?** for a discussion on this point). Moreover, most of our results can be easily extended to even more general assumptions on how players evaluate sets of (tied) alternatives. This point is explained in section 8.2.

[10] We might say "couple" intead of "society."

4 Existence of equilibrium

As previously mentioned, the focus of the paper is on pure strategy equilibria. We start with a preliminary obvious observation.

Lemma 1. *There is no equilibrium with no approved alternatives.*

Proof. If there are no approved alternatives, every alternative gets chosen with 0 votes. Then a player can vote for her preferred alternative, which will be the outcome. Because we have assumed that there are at least two different alternatives and ruled out indifferences, this is a strict improvement. **Q.E.D.**

We distinguish the equilibria according to the maximal score of the alternatives, which can be 1 or 2, given Lemma 1.

Equilibria Types. We say that an equilibrium is *consensual* when the maximal score of the alternatives equals 2. In such an equilibrium, at least one alternative is simultaneously approved by both players. An equilibrium is *non-consensual* if the maximal score equals one. In this case, no alternative is approved by both players.

Winning Sets. The winning set of a consensual equilibrium is a singleton. To see why, assume that there are several alternatives approved by both players, Then player i is then strictly better off voting only for his preferred alternative among those, making it the sole winner. (A formal proof is provided in the proof of the next theorem.) Hence in a consensual equilibrium one and only one alternative is approvoved simultaneaouly by both players and this single alternative is the outcome. Since both players vote for disjoint sets of alternatives in a non-consensual equilibrium, all the approved alternatives are in the winning set, which thus contains at least two alternatives. Proposition 3 in the Appendix provides a simple characterization of these equilibria: (B_1, B_2) is a non-consensual equilibrium iff $B_1 \cap B_2 \neq \emptyset$ and for $i = 1, 2$, $B_i = \{x \in X : u_i(x) > u_i(B_1 \cup B_2)\}$.

While traditional existence results need mixed strategies, our focus is on pure strategies. The next result proves that pure equilibria exist. The proof is original and constructive. It builds a sequence of iterated best responses and proves that at some step, both players must be playing a best response against each other. As an important by-product of the proof, we prove that there is at least one sincere equilibrium in pure strategies, i.e. an equilibrium in which both players cast sincere ballots. Indeed, in the equilibrium built in the proof, if a player approves of a candidate, she also approves of all the candidates she prefers to this candidate (see Section 6 for a discussion of sincerity).

Theorem 1. *There exists a pure strategy equilibrium in which both players are sincere.*

Proof. In the first part of the proof, we describe precisely the players' best responses. In the second part, we select a best response function for each player and define a sequence of iterated best responses. Finally, we show that this sequence at some point provides a pure strategy equilibrium. Within the proof, we use labellings of the alternatives such that:

$$u_1(a_1) > u_1(a_2) > \ldots > u_1(a_k), \text{ and } u_2(b_1) > u_2(b_2) > \ldots > u_2(b_k).$$

In other words, a_i and b_j respectively stand for the i^{th} and j^{th} preferred alternative for players 1 and 2. The notation $[x,+]_i$ stands for the upper-contour set of alternative x for player i so that

$$[x,+]_i = \{y \in X \mid u_i(y) \geq u_i(x)\}.$$

Also, in order to avoid multiple notation for utilities, we denote by $U_i(Y,Z)$ the utility achieved by player i when player 1 plays Y and player 2 plays Z. Observe that if $Y \cap Z \neq \emptyset$, the outcome is the uniform lottery on $Y \cap Z$ so that, with our notation, $U_i(Y,Z) = U_i(Y \cap Z)$ in that case. If $Y \cap Z = \emptyset$ then it equals $U_i(Y \cup Z)$, or $U_i(X)$ in the special case where $Y = Z = \emptyset$.

Part 1: Best Responses:

Let $Z \subseteq X$ denote the vote of player 2. Let $Y \subseteq X$ be a best response for player 1 to Z. There are two cases: either $Y \cap Z \neq \emptyset$ or $Y \cap Z = \emptyset$.

Case 1: $Y \cap Z \neq \emptyset$

In this case, the winning set $W(Y,Z)$ equals $Y \cap Z$, and we will now see that $Y \cap Z$ must be a singleton. Indeed, assume that there is more than one alternative in $Y \cap Z$, and let $y_0 \in Y \cap Z$ be such that $u_1(y_0) = \max_{z \in Z} u_1(z)$. Then $U_1(Y,Z) = U_1(Y \cap Z) = \frac{1}{\#Y \cap Z} \sum_{z \in Y \cap Z} u_1(z)$. However, since y_0 satisfies $u_1(y_0) > u_1(z)$ for any $z \in Y \cap Z \setminus \{y_0\}$, it follows that $U_1(y_0,Z) = u_1(y_0) > U_1(Y,Z)$, proving that Y is not a best response.

It follows that Y is of the form $Y = \{y_0\} \cup Y'$ for any $Y' \subset X \setminus Z$. In particular, notice that one can choose

$$Y = [y_0,+]_1.$$

The expected utility of player 1 equals $U_1(Y,Z) = u_1(y_0)$ in that case.

Case 2: $Y \cap Z = \emptyset$

First note that $Y = Z = \emptyset$ is impossible, because the best response to $Z = \emptyset$ is $Y = \{a_1\}$. Therefore, in case 2, the winning set $W(Y,Z)$ is equal to $Y \cup Z$.

Still denote $y_0 \in Z$ the alternative such that $u_1(y_0) = \max_{z \in Z} u_1(z)$. It must be the case that $U_1(Y,Z) = U_1(Y \cup Z) > u_1(y_0)$. Indeed, $U_1(y_0,Z) = u_1(y_0)$ and, since Y is a best response to Z, $U_1(Y,Z) > U_1(y_0,Z) = u_1(y_0)$.[11] Hence, for any $z \in Z$, we can write $U_1(Y \cup Z) > u_1(y_0) \geq u_1(z)$.

Moreover, for any $y \in Y$, $u_1(y) > U_1(Y \cup Z)$. Indeed, since Y is a best response to Z, it is a better response than $Y \setminus \{y\}$, thus

$$U_1(Y,Z) \geq U_1(Y \setminus \{y\}, Z) = U_1(Y \cup Z \setminus \{y\}).$$

Thanks to the genericity condition, the inequality is strict, and one can write:

$$\begin{aligned}
& \frac{1}{\#(Y \cup Z)} \sum_{z \in Y \cup Z} u_1(z) \; > \; \frac{1}{[\#(Y \cup Z) - 1]} \sum_{z \in Y \cup Z \setminus \{y\}} u_1(z), \\
\iff \quad & [\#(Y \cup Z) - 1] u_1(y) \; > \sum_{z \in Y \cup Z \setminus \{y\}} u_1(z), \\
\iff \quad & u_1(y) \; > \; \frac{1}{[\#(Y \cup Z)]} \sum_{z \in Y \cup Z} u_1(z) = U_1(Y \cup Z),
\end{aligned}$$

as wanted.

Similar reasoning shows conversely that, if $z \in X$ satisfies $u_1(z) > U_1(Y \cup Z)$, then $z \in Y$.

In other words, Y is uniquely defined such that:

$$Y = \{x \in X : u_1(x) > u_1(Y \cup Z)\} = [z_0, +]_1$$

for some $z_0 \in X$. In this case, $u_1(Y,Z) = U_1([z_0,+]_1 \cup Z)$. Since Y is unique, z_0 is well-defined.

Conclusion of Part 1

For any $Z \subseteq X$, we have found the best responses Y to Z for player 1. In case 1 ($Y \cap Z \neq \emptyset$), there is a set of best responses to which $[y_0,+]_1$ belongs. In the second

[11] Note that different winning sets must lead to different payoffs due to the genericity condition.

case, there is a unique best response; it equals $[z_0,+]_1$.

Building on this analysis, Part 2 defines a sequence of best responses.

Part 2: Best Response Selection and Iterated Best Responses

For any $Z \subseteq X$ played by player 2, we specify a unique best response of player 1, denoted $Y = \mathcal{R}_1(Z)$, as follows:

1. If the best responses to Z lead to some Y with $Y \cap Z \neq \emptyset$, then $\mathcal{R}_1(Z) = [y,+]_1$ with y such that $u_1(y) = \max_{z\in Z} u_1(z)$. In this case, $W(Y,Z) = \{y\}$.

2. If, on the contrary, the best response to Z leads to some Y with $Y \cap Z = \emptyset$, then $\mathcal{R}_1(Z) = \{x \in X : u_1(x) > u_1(Y \cup Z)\} = [z,+]_1$ for $z \in X \setminus Z$ defined in the text.

The same definition applies to yield a best response function $\mathcal{R}_2$ of player 2. Hence, for each player, we have selected from the best response correspondence a best response function that yields a unique response to any possible ballot of the other player.

Let $Y_0 = \{a_1\}$ (where a_1 is the preferred alternative for player 1) and, for any integer $t \geq 0$, define

$$\begin{cases} Y_{2t+1} = \mathcal{R}_2(Y_{2t}), \\ Y_{2t+2} = \mathcal{R}_1(Y_{2t+1}). \end{cases}$$

This definition generates a sequence of strategy profiles $(Y_0, Y_1), (Y_2, Y_1), (Y_2, Y_3), \ldots$. Note that Y_t stands for the strategy of player 1 if t is even and for the one of player 2 when t is odd. Part 3 proves that this sequence contains an equilibrium.

Part 3: Reaching an equilibrium

We will first prove, by induction, that non-consensual best responses are increasing:

Lemma 2. *Let $t \geq 3$. If $Y_1, Y_2, \ldots Y_t$ are t subsets of X such that $Y_1 = \{a_1\}$ and, for any $\tau \in \{2, \ldots, t\}$, Y_τ is the non-consensual best response to $Y_{\tau-1}$ for player 1 when τ is odd and for player 2 when τ is even, then $Y_{t-2} \subseteq Y_t$.*

Proof. For $t = 3$ the result is true because non-consensual best responses are sincere, which implies that Y_3 contains a_1. For $t \geq 3$, suppose that the claim holds for t and let $Y_1 = \{a_1\}, Y_2, \ldots Y_{t+1}$ be iterated non-consensual best responses. We have to prove that

$Y_{t-1} \subseteq Y_{t+1}$. From the induction hypothesis, $Y_{t-2} \subseteq Y_t$. If $Y_{t-2} = Y_t$ then $Y_{t-1} = Y_{t+1}$ because non-consensual best responses are unique, so suppose that

$$Y_{t-2} \subsetneq Y_t,$$

and suppose also, for a contradiction, that

$$Y_{t+1} \subsetneq Y_{t-1}.$$

Without loss of generality, take t even. Figure 1 represents this situation; alternatives corresponds to points in the utility space.

[Insert Figure 1 about here]

Because Y_t is the non-consensual response to Y_{t-1}, we know that

$$Y_{t-1} \cap Y_t = \emptyset.$$

For player 1, Y_{t-1} is the non-consensual best response to Y_{t-2}, thus the points in $Y_t \setminus Y_{t-2}$ are such that:

$$\forall y \in Y_t \setminus Y_{t-2}, u_1(y) < U_1(Y_{t-1} \cup Y_{t-2}).$$

Adding these points strictly decreases the payoff to player 1 so that:

$$U_1(Y_{t-1} \cup Y_t) < U_1(Y_{t-1} \cup Y_{t-2}). \qquad (a)$$

Moreover, if, as we have assumed, Y_{t+1} is a strict subset of Y_{t-1}, then

$$\forall x \in Y_{t-1} \setminus Y_{t+1}, u_1(x) > U_1(Y_{t-1} \cup Y_{t-2}),$$

so that we obtain, combining the previous observation with (a):

$$\forall x \in Y_{t-1} \setminus Y_{t+1}, u_1(x) > U_1(Y_{t-1} \cup Y_t).$$

Removing the points x from Y_{t-1} can thus only decrease the average, hence:

$$U_1(Y_{t+1} \cup Y_t) < U_1(Y_{t-1} \cup Y_t),$$

in contradiction with the fact that Y_{t+1} is the best response to Y_t. This completes the proof of the lemma. $\square$

To complete the proof of the theorem, consider the sequence of strategy profiles $(Y_1, Y_2), (Y_3, Y_2), (Y_3, Y_4), \ldots$ defined above.

If $Y_t \cap Y_{t+1} = \emptyset$ for all t, the previous lemma dictates that the sequences $(Y_{2s})_{s \in I\!N}$ and $(Y_{2s+1})_{s \in I\!N}$ are increasing in s. Since the number of alternatives is finite, they must be stationary from some period onwards. Hence there must exist an equilibrium in pure strategies since from this period onwards both players are playing a best response to each other's strategy.

If $Y_\tau \cap Y_{\tau+1} \neq \emptyset$ for at least some τ, take the smallest such τ and denote it by t:

$$Y_t \cap Y_{t+1} \neq \emptyset.$$

If $t = 1$, recall that $Y_1 = \{a_1\}$, then $Y_{t+1} \cap Y_t \neq \emptyset$ just means that $a_1 \in Y_2$, which implies that the same alternative $a_1 = b_1$ is preferred by both players. Then $(\{a_1\}, \{a_1\})$ is an equilibrium.

If $t > 1$ then

$$Y_{t-1} \cap Y_t = \emptyset$$

by definition of t. Without loss of generality, suppose that t is even, so that Y_t is played by player 2 and Y_{t-1} and Y_{t+1} are played by player 1. (The same claims hold if t is odd by exchanging the roles of the players.) Hence Y_{t+1} is a consensual best response for player 1 to Y_t.

We represent the argument in Figure 2 in which, again, alternatives are represented by points in the utility space.

[Insert Figure 2 about here]

We know that we can write for some i: $Y_{t+1} = [a_i, +]_1$, with

$$\{a_i\} = Y_t \cap Y_{t+1}.$$

To prove that (Y_{t+1}, Y_t) is an equilibrium, we just have to prove that Y_t is a best response for player 2 to Y_{t+1}.

Because $a_i \in Y_t$ and $Y_t \cap Y_{t-1} = \emptyset$, $a_i \notin Y_{t-1}$, thus

$$Y_{t-1} \subsetneq [a_i, +]_1.$$

Consider x in $[a_i,+]_1 \setminus Y_{t-1}$, with $x \neq a_i$. Then x cannot belong to Y_t, for a_i is the best point in Y_t for player 1. Define $u_2^* = U_2(Y_{t-1} \cup Y_t)$. Since Y_t is a non-consensual best response, $x \notin Y_t$ implies that $u_2(x) < U_2(Y_{t-1} \cup Y_t) = u_2^*$. It follows that the payoff to player 2 decreases when adding these alternatives x from Y_{t-1} to $Y_{t+1} \setminus [a_i,+]_1$ so that:

$$U_2([a_{i-1},+]_1 \cup Y_t) \leq u_2^*.$$

Now if the best response of player 2 to $[a_i,+]_1$ is consensual, it cannot yield more than $u_2(a_i)$, which is achieved by Y_t.

If the best response is non-consensual, it must yield more than $u_2(a_i)$ and thus be of the form $[b_j,+]_2$, with $u_2(b_j) > u_2(a_i)$. However, then we have:

$$u_2(a_i) < U_2([a_i,+]_1,[b_j,+]_2) = U_2([a_i,+]_1 \cup [b_j,+]_2) \leq U_2(Y_{t-1} \cup [b_j,+]_2).$$

(Because the alternatives in $[a_i,+]_1 \setminus Y_{t-1}$ have a payoff of at most $u_2(a_i)$, player 2 prefers them to be removed.) It follows that

$$u_2^* < u_2(a_i) \leq U_2(Y_{t-1},[b_j,+]_2),$$

in contradiction with the fact that the best response to Y_{t-1} yields u_2^*.

We conclude that the consensual response Y_t is optimal. It follows that (Y_{t+1}, Y_t) is an equilibrium. **Q.E.D.**

5 Random Lower Bound

Once we have established the existence of pure strategy equilibria, we focus on the properties satisfied by the Approval Mechanism.

We start by considering the lowest expected utility a player might get in equilibrium. As will be observed below, in any equilibrium, each player gets at least his mean (and his median) expected utility.

We define the *average outcome* as the one given by the tie among all the candidates, i.e. the winning set X. For $i = 1, 2$, denoting $k = \#X$:

$$U_i(X) = \frac{1}{k} \sum_{x \in X} U_i(x).$$

The following condition is the natural expected utility version of equal division lower bound and is often present in the literature in fair allocation rules.

Random Lower Bound (RLB) : An outcome $W \subseteq X$ satisfies Random Lower Bound if it gives at least the mean outcome to both players.

This condition is rather mild and intuitive. Indeed, randomness is often used as a device for making a "fair" choice when discriminating between alternatives in the absence of objective reasons.

Yet, as has been shown by the discussion in Section 2, no deterministic procedure that assigns an alternative to any utility vector can satisfy it.

On the contrary, the Approval Mechanism satisfies *RLB* in any voting situation. In order to show this claim, we introduce the notion of uniform sincerity, extensively discussed in the early works on Approval Voting (see Merill [1979], Merill and Nagel [1987] and references therein). It can be defined as follows:

Uniform Sincerity: A player's choice is uniformly sincere if she votes for precisely those candidates whose utility exceeds the average. More formally, for each player i, the uniformly sincere ballot is

$$S_i = \{x \in X \mid u_i(x) > U_i(X)\}.$$

Uniform Sincerity is optimal for a player with a uniform belief on the other players' strategies (Ballester and Rey-Biel [2009]). In our setting, the purely sincere ballot plays the role of a benchmark since it ensures that the player gets at least the mean utility.

Theorem 2. *For any pure strategy of the opponent, uniform sincerity delivers to the player at least her mean utility.*

Theorem 2, the proof of which is included in the appendix, directly implies that any equilibrium must satisfy Random Lower Bound, and hence we state the following corollary without proof.

Corollary 1. *The Approval Mechanism satisfies Random Lower Bound.*

Notice that this very simple result does not hold for more than two players. For instance, if two players out of three agree on an alternative, this may be an equilibrium outcome even if it is very detrimental to the third player.

As usual in fair division problems, it is not obvious whether requiring the mean outcome for every player is more appealing than requiring the median one. As we now show, this problem is absent from our setting since the median outcome is equivalent to the mean one.

Formally, for any $W \subseteq X$ and some player i, we let $\mathcal{L}_i(W)$ and $\mathcal{H}_i(W)$ respectively denote the number of outcomes (i.e. uniform lotteries) that deliver the player i lower and higher expected utility than W. Formally, $\mathcal{L}_i(W) = \#\{V \subseteq X \mid U_i(V) < U_i(W)\}$ and $\mathcal{H}_i(W) = \#\{V \subseteq X \mid U_i(V) > U_i(W)\}$.

We define the *median outcome* for player i as the one given by the outcome W with $\mathcal{L}_i(W) = \mathcal{H}_i(W)$. As we consider the set of outcomes are the uniform lotteries, we now prove that the median outcome coincides with the mean one.

Lemma 3. *The Mean outcome equals the Median one.*

Proof. By definition, the mean outcome is the tie among all candidates in X. Hence, its expected utility equals $U_i(X)$ with $U_i(X) = \frac{1}{k}\sum_{x\in X} u_i(x)$ for each player i. Assume that player i prefers the victory of some subset W of size $j < k$ to the one of X so that

$$\frac{1}{j}\sum_{x\in W} u_i(x) > \frac{1}{k}\sum_{x\in X} u_i(x) \iff k\sum_{x\in W} u_i(x) > j\sum_{x\in X} u_i(x). \tag{2}$$

Simple algebra proves that (2) is equivalent to $(k-j)\sum_{x\in W} u_i(x) > j\sum_{x\in X\setminus W} u_i(x)$. Consider now that player i prefers X over $X \setminus W$ so that:

$$\frac{1}{k}\sum_{x\in X} u_i(x) > \frac{1}{k-j}\sum_{x\in X\setminus W} u_i(x) >\iff (k-j)\sum_{x\in X} u_i(x) > k\sum_{x\in X\setminus W} u_i(x). \tag{3}$$

Again simple algebra proves that (3) is equivalent to $(k-j)\sum_{x\in W} u_i(x) > j\sum_{x\in X\setminus W} u_i(x)$. Hence, as (2) and (3) are equivalent, it follows that player i prefers some winning set W to X if and only if she prefers X to $X \setminus W$.

Note that this implies that $\mathcal{L}_i(X) = \mathcal{H}_i(X)$ since whenever a player prefers some set to X, she also prefers X to its complementary (note that we have assumed a player is never indifferent between two different winning sets). In other words, the complete tie delivers the median expected utility, as wanted. **Q.E.D.**

One could argue that the axiom of Random Lower Bound is quite mild since it is trivial to satisfy. For instance, one can consider simultaneous or dynamic mechanisms with an outside option for both players which generates the lottery over X.

However, is it possible to design mechanisms that give strictly more utility to both players than the lottery over X *for every utility profile*? As will be shown now, this is not possible. In this sense, the random lower bound is the highest minimal utility level for both players one can design.

Let $\mathcal{L}(X)$ denote the set of all uniform lotteries over X. A social choice correspondence (SCC) f is a mapping $f : \mathcal{U}_1 \times \mathcal{U}_2 \to \mathcal{L}(X)$. For each $u = (u_1, u_2)$, $f(u) \subseteq \mathcal{L}(X)$ denotes the outcome of the SCC.

Lemma 4. *There is no SCC than can ensure both players a utility strictly higher than $U(X)$.*

Proof. Let $k = \#X$ denote the number of alternatives. Let u_1 denote the utility vector for player 1. Let u_2 such that $u_2(j) = u_1(k-j)$ for each $j \in \{1, \ldots, k\}$. Hence, the preferences are perfectly opposed. As argued by Lemma 3, the lottery over all the elements of X is the median outcome. Assume that we require a social choice correspondence to give strictly more utility than $U(X)$ to player 1. Since the preferences are perfectly opposed and $U(X)$ denotes the median outcome, any lottery that gives strictly more utility than $U(X)$ to player 1 must give less utility than $U(X)$ to player 2, proving the claim. **Q.E.D.**

6 Partial Honesty and Sincerity

We now evaluate whether the approval game triggers sincerity. As usual in the study of approval voting, a strategy is *sincere* if, given the lowest-ranked alternative that a player approves of, she also approves of all alternatives ranked higher (see Brams and Fishburn [1983-2007] and Laslier and Sanver [2010]). Formally,

Definition 1. *A ballot B is sincere for a player i if* $(u_i(x) > u_i(y) \ \textit{ and } \ y \in B) \Rightarrow x \in B$.

The set of sincere ballots for player i is denoted by $\mathcal{S}(\mathcal{B}_i)$. Note that this sincerity notion is rather weak since a player can hesitate between several such ballots, nevertheless, the next example proves that the players need not approve their most preferred alternative in equilibrium.

Example 1: Let $X = \{a, b, c, d, e\}$ with:

$$u_1 = (100, 0, 90, 20, 40), \text{ and } u_2 = (100, 0, 90, 40, 20).$$

There is an equilibrium $(ce; cd)$ in which neither of the players vote for their most preferred alternative. To see why, consider player 1's decision problem. Given that 2 votes cd, player 1 needs to determine the ballot B_1 that maximizes his utility. Given the different ballots $B_1 \in \mathcal{B}_1$, he might obtain the following winning sets:

$$\{c, d, cd, acd, bcd, cde, abcd, acde, bcde, abcde\},$$

which respectively deliver the expected utilities:

$$\{90, 20, 110/2, 210/3, 110/3, 150/3, 210/4, 250/4, 150/4, 250/5\}.$$

His best response is then to pick a ballot that leads to the election of c; ce is hence a best response. Since the players have symmetric preferences (with a switch between d and e), the same argument proves the claim for player 2, showing that $(ce; cd)$ is an equilibrium.

In the remainder of the paper, we assume that in case of an equal outcome, individuals prefer to bargain by reporting those alternatives that provide more utility than the least approved one.

As will be shown, the very mild assumption of partial honesty is enough to restore sincere best responses. In order to define this behavioral assumption, we denote by $\succeq_i$ the individual's ordering over the strategy profiles (B_i, B_{-i}) in $\mathcal{B}$. Its asymmetric component is denoted by $\succ_i$.

Definition 2 (Dutta and Sen [2012]). *A player i is partially honest whenever for all (B_i, B_{-i}), $(B'_i, B_{-i}) \in \mathcal{B}$.*

(i) *If $U_i(B_i, B_{-i}) \geq U_i(B'_i, B_{-i})$ and $B_i \in \mathcal{S}(\mathcal{B}_i)$, $B'_i \notin \mathcal{S}(\mathcal{B}_i)$, then $(B_i, B_{-i}) \succ_i (B'_i, B_{-i})$.*

(ii) *In all other cases, $(B_i, B_{-i}) \succeq_i (B'_i, B_{-i})$ iff $U_i(B_i, B_{-i}) \geq U_i(B'_i, B_{-i})$.*

The first part of the definition represents the individual's partial preference for honesty — he strictly prefers the strategy profile (B_i, B_{-i}) to (B'_i, B_{-i}) when he plays sincerely in (B_i, B_{-i}), but not in (B'_i, B_{-i}) provided the outcome corresponding to (B_i, B_{-i}) is at least as good as that corresponding to (B'_i, B_{-i}). The second part of the definition is standard.

The preference profile $(\succeq_1, \succeq_2)$ now defines a modified normal form game. We omit formal definitions for the sake of brevity. Note that Theorem 1 ensures the existence of an equilibrium of the original game in which players's strategies are sincere (by construction). Therefore the same strategy profile is also an equilibrium of the modified game, and we can state without further proof:

Theorem 3. *The game with partially honest players has a pure strategy equilibrium.*

Notice that the sincerity property is in fact true not only in the equilibrium we previously built but in any pure equilibrium of the modified game.

Theorem 4. *In any pure strategy equilibrium of the game with partially honest players, all the players use sincere strategies.*

Proof. It is well known that under Approval Voting, for every pure strategy of the other players, the set of pure best replies contains a sincere best response (see De Sinopoli et al. [2006] and Endriss [2013]). Hence, in any pure strategy equilibrium, every player is indifferent between casting a honest ballot or a dishonest one. Since the players are assumed to be partially honest, the result follows. **Q.E.D.**

7 Pareto Efficiency

We now consider the Pareto properties of the approval equilibria. Compared to most of the theoretical bargaining literature, our setting is unusual in the sense that it only considers *uniform* lotteries among the different alternatives.

Our analysis focuses on three main aspects: the Lottery Pareto Efficiency, the milder of Pure Pareto Efficiency and finally the results dealing with implementation theory.

Lottery Pareto Efficiency

An outcome $W \subseteq X$ Pareto dominates an outcome $V \subseteq X$, if $U_i(W) > U_i(V)$ for both $i = 1, 2$. Hence, the most intuitive notion of Pareto dominance is as follows:

Lottery Pareto Efficiency (LPE) : An outcome W is *Lottery Pareto Efficient* if it is not Pareto dominated by a uniform lottery.

The Approval mechanism encompasses some notion of Pareto efficiency. Indeed, as will be shown by Theorem 5, every equilibrium satisfies the strong notion of *LPE* as long as there are just three alternatives.

Theorem 5. *Let $k = 3$. Every equilibrium satisfies Lottery Pareto Efficiency.*

The proof is included in the Appendix. It is constructive and describes for each possible vector u, the set of equilibria outcomes. Yet, this positive result does not extend to any number of alternatives. Indeed, in the next example with four alternatives, the unique equilibrium does not satisfy Lottery Pareto Efficiency.

Example 2: Let $X = \{a, b, c, d\}$ and assume that the utility vector is:

$$u_1 = (100, 75, 45, 0) \ u_2 = (0, 45, 75, 100).$$

In this society, the outcome bc Pareto dominates the outcome $abcd$. Indeed, the former one gives an expected utility of 60 to each player whereas the second one gives 55. However, $B = (ab; cd)$, yielding the outcome $abcd$ is the unique equilibrium of the game. **Q.E.D.**

Pure Pareto Efficiency

Yet, even if *LPE* does not hold, some notion of Pareto optimality is incorporated in the Approval Mechanism. As will be shown, under the assumption of partial honesty, all equilibria satisfy the next notion of Pareto efficiency: it entails that there is no alternative that Pareto dominates the winning set.

Pure Pareto Efficiency (PPE): An outcome W is *Pure Pareto Efficient* if it is not Pareto dominated by any pure alternative.

Theorem 6. *With partially honest players, every equilibrium satisfies Pure Pareto Efficiency.*

The proof is structured in the Lemmata 5,6 and 7. In each of these Lemmata, it is assumed that both players are partially honest.

Lemma 5. *In a unanimous society, the winning set satisfies Pure Pareto Efficiency.*

Proof. Let a denote the unanimously preferred alternative. It follows that a is included in any best response of the players. Therefore, a wins in any equilibrium, proving the claim. **Q.E.D.**

Lemma 6. *In a non-unanimous society, a non-consensual equilibrium satisfies Pure Pareto Efficiency.*

Proof. Let $B = (B_1, B_2)$ be a non-consensual equilibrium. Assume that there is some $y \in X$ for which $u_i(y) > U_i(B_1, B_2) = U_i(B_1 \cup B_2)$ for both $i = 1, 2$. Lemma 3 entails that $B_1 \cap B_2 = \emptyset$, and for any player $i \in \{1, 2\}$, $B_i = \{x \in X : u_i(x) > U_i(B_1 \cup B_2)\}$. Therefore, the score of y equals 2, which contradicts B being a non-consensual equilibrium, proving the claim. **Q.E.D.**

Lemma 7. *In a non-unanimous society, a consensual equilibrium satisfies Pure Pareto Efficiency.*

Proof. Suppose by contradiction that the winning set of a consensual equilibrium B is not Pure Pareto Efficient. As previously argued, this winning set is a singleton which we denote by x. Since x is not Pareto efficient, there is some $y \in X$ such that $u_i(y) > u_i(x)$ for both $i = 1, 2$. Moreover, since the score of x equals 2, both players vote for x. There are three possibilities for the score of y given B, denoted $s_y(B)$.

Assume first that $s_y(B)$=0. No player has voted for y. Hence, since $s_x(B) = 2$, adding one point to y does not modify the winning set. Since each player is partially honest, she must approve of y, contradicting $s_y(B) = 0$.

Assume now that $s_y(B)$=1. W.l.o.g. assume that player 1 has approved of y whereas player 2 has not. If player 2 modifies his pure strategy and ceases to vote for x and votes for y, then y is the unique element in the winning set. Since $u_2(y) > u_2(x)$, B is not an equilibrium, since player 2 is not playing a best response.

Finally, assume that $s_y(B) = 2$. Then there are at least two alternatives in the winning set of this consensual equilibrium. But we already noticed that in a consensual equilibrium, the winning set is a singleton.

Therefore, the winning set of a consensual equilibrium satisfies Pure Pareto Efficiency. **Q.E.D.**

Implementation Theory

As will be shown, there is no hope of implementing in pure strategies a social choice correspondence that satisfies together the axioms of Random Lower Bound and Pure Pareto Efficiency. The tension between implementability in pure strategies and Pure

Pareto Efficiency was emphasized earlier by Hurwicz and Schmeidler [1978] and Maskin [1999], but their results do not directly imply the described impossibility, because our mechanism applies to the specific context in which the game outcomes are uniform lotteries over the set of the alternatives. More recently, Dutta and Sen [2012] proved that the Pareto correspondence itself is not implementable when both players are partially honest.

Let $\mathcal{L}(X)$ denote the set of all uniform lotteries over X. A social choice correspondence (SCC) f is a correspondence $f : \mathcal{U}_1 \times \mathcal{U}_2 \to \mathcal{L}(X)$. For each $u = (u_1, u_2)$, $f(u) \subseteq \mathcal{L}(X)$ denotes the outcome of the SCC.

Definition 3. *A SCC f is fully implementable if there exists a game form Γ such that for each $u \in \mathcal{U}_1 \times \mathcal{U}_2$, all the pure-strategy Nash equilibria of (Γ, u) have the same outcome, and this outcome is $f(u)$.*

Proposition 1. *There is no SCC that is implementable in pure strategies that satisfies Random Lower Bound and Pure Pareto Efficiency.*

Proof. The proof extends the one of De Clippel et al. [2013] to cardinal utilities. Let $u = (u_1, u_2)$ be a utility profile on $\{a, b, c, d, e\}$ with $u_1 = (100, 53, 52, 1, 0)$ and $u_2 = (0, 1, 52, 53, 100)$. Players 1 and 2 have opposed preferences. Moreover, if a $RSCF$ satisfies RLB and LPE, it must be the case that $f(u) = c$ since it is the unique lottery which satisfies both desiderata. Then, denoting by $v_1 = (52, 1, 100, 0, 53)$ and $v_2 = (52, 1, 53, 0, 100)$, Maskin Monotonicity (a necessary condition for implementation in pure strategies) implies that $f(v) = c$ for $v = (v_1, v_2)$.

We now consider the profile $w = (w_1, w_2)$ with $w_1 = (53, 1, 100, 0, 52)$ and $w_2 = (1, 53, 0, 100, 52)$. The conditions of RLB and LPE jointly imply that $f(w) = e$. Moreover, since f satisfies Maskin monotonicity, it must be the case that $f(v) = e$. But this entails a contradiction with $f(v) = c$, proving the claim. **Q.E.D.**

Proposition 2. *Assume that both players are partially honest. There is no SCC that is implementable in pure strategies and satisfies Random Lower Bound and Pure Pareto Efficiency.*

Proof. Proposition 1 in Dutta and Sen [2012] implies no SCC which contains the union of the best-ranked alternatives of the two players is implementable even when both individuals are partially honest. As shown by the introductory example in

Section 2, there exists preference profiles (when preferences are perfectly opposed and for some specification of cardinal utilities) for which *RLB* and *PPE* single out the union of the best-ranked alternatives as the unique acceptable lottery proving the claim. **Q.E.D.**

8 Variations

This section focuses on two intuitive variations on the current work. We first consider the role of mixed strategies and show that while the property of Random Lower Bound still holds with mixed strategies, the incentives for sincerity may disappear if both players randomize. The second variation extends the framework in two directions. Building on the literature on preferences over sets (see **?**), we allow any tie-breaking rule and we also relax the assumption that the players are expected utility maximizers. We then give sufficient conditions over preferences over sets to ensure that each player has always a sincere best response and at the same, the game admits a pure strategy equilibrium.

8.1 Mixed strategies

Up to now, we have focused on pure strategy equilibria. In this section, we analyze whether the previous properties of the Approval Mechanism are still valid when mixed strategy equilibria are also taken into account.

Random Lower Bound The property of Random Lower Bound still holds in any mixed strategy equilibrium, because a player can always play the pure sincere strategy that ensures to her at least her mean utility (Theorem 2). We thus can state:

Theorem 7. *The Approval Mechanism satisfies Random Lower Bound in mixed strategies.*

Sincerity. The next example will prove that there might exist equilibria in which one of the players strictly prefers to play insincerely. Note that the violation of Sincerity must be in mixed strategies, since in pure strategies, the set of best responses always includes a sincere one (and hence partial honesty removes insincere equilibria in pure strategies).

Let $X = \{a, b, c, d, e\}$ and assume that the players' utilities are as follows:

$$u_1 = (100, 75, \frac{200}{3}, \frac{175}{3}, 0), \text{ and } u_2 = (65, \frac{3200}{49}, 35, 0, 100).$$

Consider the strategy profile σ in mixed strategies with

$$\sigma_1(ac) = 0.9, \text{ and } \sigma_1(acd) = 0.1,$$

and

$$\sigma_2(e) = 0.15,\ \sigma_2(be) = 0.25 \text{ and } \sigma_2(abe) = 0.6.$$

In this profile, player 1 does not respect sincerity since the two ballots in his support are insincere in the sense that they skip candidate b. Moreover, player 1 votes for his preferred alternative and for c and d, the least two preferred alternatives of player 2. This creates an uncertainty for player 2 that leads him to mix among three of his sincere strategies.

As will now be shown, σ is an equilibrium, so that we can state the next result.

Theorem 8. *Insincerity might be a strict best response in equilibrium when players use mixed strategies.*

To understand the intuition, consider the next table, in which the pure strategies of player 1 are the rows, whereas thoses of player 2 are the columns.

	e	be	abe
ac	ace	$abce$	a
acd	$acde$	$abcde$	a
ab	abe	b	ab

In the table are represented the winning sets so that, for instance, the winning set that corresponds to the strategy pair (ac, e) (first column, first row) is ace. Since $U_1(acd, e) > U_1(ac, e)$ and $U_1(acd, be) < U_1(ac, be)$, it follows that player 1 is indifferent between playing ac and acd when player 2 plays σ_2. Indeed, the figures are such that

$$\sigma_2(e)\frac{(100 + \frac{200}{3})}{3} + \sigma_2(be)\frac{(175 + \frac{200}{3})}{4} = \sigma_2(e)\frac{(100 + \frac{375}{3})}{4} + \sigma_2(be)\frac{(175 + \frac{375}{3})}{5}.$$

A symmetric argument applies to player 2 when player 1 plays σ_1. Note that, in equilibrium, player 2 is indifferent between his three strategies in the support: e, be

and abe. Since abe leads to the victory of a given σ_1, this implies that

$$U_2(\sigma_1, be) = U_2(\sigma_1, abe) = u_2(a) = 65.$$

To see why this the case, note that

$$U_2(\sigma_1, be) = \frac{9}{10}\frac{200}{3} + \frac{1}{10}\frac{200}{4} = 65,$$

and

$$U_2(\sigma_1, abe) = \frac{9}{10}\frac{200 + b^*}{4} + \frac{1}{10}\frac{200 + b^*}{5} = 65,$$

with $b^* = \frac{3200}{49}$.

The rest of the proof is included in Appendix **C**.

Partial honesty does not remove this equilibrium. Indeed, it can be shown that $U_1(ac, \sigma_2) > U_1(B_1, \sigma_2)$ for any sincere ballot B_1. For instance, take the pure strategy ab. We have that $U_1(ab, e) > U_1(ac, e)$ and $U_1(ab, be) > U_1(ac, be)$. However, since $U_1(ab, abe) < U_1(ac, abe)$, player 2 playing σ_2 ensures that player 1 prefers ac to ab and to any other sincere ballot.

8.2 Alternative Tie-Breaking Rules

This last section addresses an extension of this work that deals with the tie-breaking rule. So far, we have used the uniform tie-breaking rule: whenever two or more alternatives were tied, we have assumed that one of them is selected at random. Arguably in our context, this seems to be "the most natural choice for a tie-breaking mechanism" (Endriss [2013]), and, for a more formal justification, one could refer to the axiomatization of uniform expected utility due to **?**.

Yet, it seems important to understand whether our results depend on this assumption. The mechanism first takes a pair of subsets of alternatives, merges them into an outcome which is either the intersection (if the intersection is non empty) or the union (if the intersection is empty). The outcome can be any subset of alternatives. Then the players evaluate possible outcomes by computing expected utility under uniform rule. This combines a tie-breaking rule (a single alternative is chosen at random uniformly) and a behavioral assumption (expected utility maximization). Indeed, there is a vast literature in social choice theory that extends preferences over alternatives to preferences over sets of alternatives (see **?**). We now prove that our results would be not modified if we depart from the uniform-tie breaking rule and/or allow the players not be expected-utility maximizers.

For any pair of alternatives, let xR_iy denote that i weakly prefers x to y and let P denote the strict preference: xP_iy if xR_iy and not yR_ix.

We now consider individual preferences directly for sets of alternatives. That is we do not anymore assume that each player derives his preferences over game outcomes by computing his expected utility over this set. Note that any tie-breaking rule compatible with our assumptions is a-priori allowed.

A preference over sets of alternatives is a weak order $\trianglerighteq$ on 2^X. The binary relation $\trianglerighteq$ is assumed to be complete, reflexive and transitive. We write $A \triangleright_i B$ if $A \trianglerighteq_i B$ but not $B \trianglerighteq_i A$ and $A -_i B$ if both $A \trianglerighteq_i B$ and $B \trianglerighteq_i A$. The weak order $\trianglerighteq_i$ may depend on the preferences over alternatives R_i but also on the beliefs of player i over the tie-breaking rule. Following the previous genericity assumption, which ruled out indifferences, we impose that for any pair $A, B \in 2^X$ and any $i = 1, 2$,

$$\text{either} A \triangleright_i B \,\text{or}\, B \triangleright_i A.$$

The preference profile $(\trianglerighteq_1, \trianglerighteq_2)$ now defines a modified normal form game, best responses being defined with respect to the relation $\trianglerighteq$. For any strategy Z of player j, the set of best responses of player i is:

$$\mathfrak{K}_i(Z) = \{Y \in 2^X \mid W(Y,Z) \ \trianglerighteq_i \ W(Y',Z) \,\text{for any}\, Y' \in 2^X\}.$$

We now recall some principles for extending preferences over alternatives to preferences over sets of alternatives.[12] For any set $A \in 2^X$, we write $\max_i(A)$ to denote the set maximal alternatives in A for player i. Hence:

$$\max_i(A) = \{x \in X \mid xP_iy \,\text{for any}\, y \in X\}.$$

Kelly Principle: For any $A \in 2^X$ and any $x, y \in X$, we have:

IND $\{x\} \triangleright_i \{y\}$ if xP_iy.

[12] Endriss [2013] gives conditions for the existence of a sincere best response under Approval voting under different preference extensions. More specifically, he proves that this is the case if a player's preferences over sets of alternatives follows the Kelly Principle, the ADD axioms and some replacement axiom à la Sen.

MAX $\{\max_i(A)\} \trianglerighteq_i A$.

Addition-Deletion Axioms. For any $A \in 2^X$ and any $x \in X$,

ADD $A \setminus \{x\} \triangleright_i A$ if $A \triangleright_i \{x\}$.

DEL $A \cup \{x\} \triangleright_i A$ if $\{x\} \triangleright_i A$.

We now prove that if the player's preferences satisfy the Kelly Principle and the Addition-Deletion axioms, then the game with extended preferences admits a pure strategy equilibrium and, moreover, there is always a sincere best response. This implies that our results are not altered if we replace the tie-breaking rule and the assumption of expected utility maximization with the previously described requirements.

Theorem 9. *Assume that the players' preferences satisfy the Kelly Principle and the Addition-Deletion axiom. Then in the game with preferences* $(\trianglerighteq_1, \trianglerighteq_2)$*:*

1. *a player always has a sincere best response.*

2. *there exists a pure strategy equilibrium in sincere strategies.*

Proof.

1. The proof is done for player 1 w.l.o.g. Let Z denote player 2's strategy for some $Z \in 2^X$. The player has two classes of best responses: consensual and non-consensual ones.

Consensual Best Responses: Consider first the consensual best responses. Any such best response is some set $Y \in \mathcal{R}_1(Z)$ with $Y \cap Z \neq \emptyset$ and $W(Y,Z) = Y \cap Z$.

Let $\{y_0\} = \max_1(Z)$. Note that the strategy $[y_0,+]_1$ is sincere and leads to the outcome $W([y_0,+]_1, Z) = \{y_0\}$. We claim that any consensual best response must lead to the same outcome than $[y_0,+]_1$, proving that this strategy is a sincere best response.

Assume by contradiction that there is some consensual best response Y with $W(Y,Z) = Y \cap Z \neq \{y_0\}$.

If $y_0 \in Y \cap Z$, then, by MAX, $\{y_0\} \trianglerighteq_1 Y \cap Z$, proving that $[y_0,+]_1$ is a best response.

If $y_0 \notin Y \cap Z$, then let $\{x_0\} = \max_1(Y \cap Z)$. By MAX, it follows that $\{x_0\} \trianglerighteq_1 Y \cap Z$. Moreover, by IND, $\{y_0\} \triangleright_1 \{x_0\}$. Hence, by transitivity, $\{y_0\} \triangleright_1 Y \cap Z$, proving that Y is not a best response.

This concludes the proof for the consensual best responses.

Non-Consensual Best Responses: Consider now the non-consensual best responses. A non-consensual best response is some $Y \in \mathcal{R}_1(Z)$ with $Y \cap Z = \emptyset$ and $W(Y,Z) = Y \cup Z$.

Due to ADD, for any alternative y with $\{y\} \triangleright_1 Z$, we have $Z \cup \{y\} \triangleright_1 Z$.

Moreover, due to DEL, for any alternative y with $Z \triangleright_1 \{y\}$, we have $Z \setminus \{y\} \triangleright_1 Z$.

Combining the previous observations leads to show that the unique non-consensual best response Y satisfies:

$$Y = \{y \in X \mid \{y\} \triangleright_1 Z\}.$$

Since Y is sincere, this concludes the proof for the part 1.

2. The existence proof basically hinges on the preferences of the players over the winning sets. To see that the claim is correct, it suffices to see that the the Kelly Principle and the Addition-Deletion axiom are enough to compare all the different winning sets that appear in the proof. We omit the proof of this result for the sake of brevity[13]. **Q.E.D.**

9 Conclusion

We have introduced the Approval Mechanism, a two-person dispute-resolution device, which has the virtue of being very simple to implement and exhibits three main appealing properties: First, the existence of a pure strategy equilibrium in which both players play sincere strategies. Second, the flexibility of the mechanism ensures that both players get at least their mean (and median) expected utility in every equilibrium. This is a basic requirement in dispute resolution settings like ours, but it cannot be satisfied by a scoring rule nor by a deterministic mechanism. Third, any equilibrium satisfies Lottery Pareto Efficiency with three alternatives and Pure Pareto Efficiency for any arbitrarily large number of alternatives. As has been shown, the tension between the existence of a pure strategy equilibrium and Pure Pareto Efficiency is not absent from our setting, as previously emphasized by the Nash implementation literature. However, while the Approval mechanism partially implements the set of lotteries satisfying both *PPE* and *RLB*, we prove that no mechanism can fully implement this set.

[13] A self-contained proof is available in the on-line appendix.

Our device is hence particularly well-fitted to ensure a minimal degree of impartiality while taking into account the players' intensities of preferences. Ours is hence a pioneering work in applying a voting device to a dispute resolution environment. Note that the properties of this mechanism depend on the environment with two players, since each player has more "power" with respect to the final outcome in the current case than in a large election. For instance, the property of Random Lower Bound is a consequence of the setting with just two-players.

The counterintuitive results in mixed strategies (as far as sincerity is concerned) confirm that the pure strategy equilibria are the more reasonable framework to theoretically predict the players' reactions to the Approval Mechanism. Indeed, one may doubt that the probability distributions over the outcomes that might arise in a mixed strategy equilibrium accurately represent the players' beliefs when facing uncertainty over the opponent's strategy.

Finally, understanding the properties of the current mechanism under incomplete information over the players' preferences seems to be a potentially interesting venue of research.

A Appendix: A Characterization of Non-Consensual Equilibria

Proposition 3. *For a non-unanimous society, a strategy profile $B = (B_1, B_2)$ is a non-consensual equilibrium if and only if $B_1 \cap B_2 = \emptyset$ and for any player $i \in \{1, 2\}$,*

$$B_i = \{x \in X : u_i(x) > U_i(B_1 \cup B_2)\}.$$

Proof. 1.$\Longrightarrow$) Let $B = (B_1, B_2)$ be a non-consensual equilibrium of the game. Then it follows that $B_1 \cap B_2 = \emptyset$, as otherwise some alternative would have a score of 2 contradicting the fact that B is a non-consensual equilibrium. Moreover, assume, by contradiction, that there is some i for which some $y \in B_i$ satisfies $u_i(y) < U_i(B_1 \cup B_2)$. Let $j = \#\{B_1 \cup B_2\}$. The previous inequality is equivalent to: $u_i(y) < \frac{1}{j} \sum_{x \in B_1 \cup B_2} u_i(x) \Longleftrightarrow (j-1) u_i(y) < \sum_{x \in B_1 \cup B_2 \setminus \{y\}} u_i(x)$. However it is simple to see that the previous inequality is equivalent to:

$$(j-1) \sum_{x \in B_1 \cup B_2} u_i(x) < j \sum_{x \in B_1 \cup B_2 \setminus \{y\}} u_i(x) \Longleftrightarrow \frac{1}{j} \sum_{x \in B_1 \cup B_2} < \frac{1}{j-1} \sum_{x \in B_1 \cup B_2 \setminus \{y\}} u_i(x).$$

In other words, $u_i(y) < U_i(B_1 \cup B_2)$ if and only if $U_i(B_1 \cup B_2) < U_i(B_1 \cup B_2 \setminus \{y\})$. However, since $y \in B_i$, the previous inequality proves that i is not playing a best response, proving that B is not an equilibrium.

2.$\Longleftarrow$) Let $B = (B_1, B_2)$ be a strategy profile. Assume that $B_1 \cap B_2 = \emptyset$ and for any player $i \in \{1, 2\}$ and any alternative $x \in X$, $B_i = \{x \in X : u_i(x) > U_i(B_1 \cup B_2)\}$. Since $B_1 \cap B_2 = \emptyset$, then the maximum score of an alternative equals one. Furthermore, any $x \in B_i$ satisfies $u_i(x) > U_i(B_1 \cup B_2)$. As previously shown, for any alternative $y \in X$, $u_i(y) < U_i(B_1 \cup B_2)$ is equivalent to $U_i(B_1 \cup B_2) < U_i(B_1 \cup B_2 \setminus \{x\})$. Hence, a player 's best response equals all the alternatives $x \in X$ such that $u_i(x) > U_i(B_1 \cup B_2)$, which coincides with B_i. Therefore, B is a non-consensual equilibrium, proving the claim.

Q.E.D.

B Appendix: Proof of Theorem 2

Suppose that player j plays some pure strategy denoted B_j. The expected utility for player i when he plays the uniformly sincere ballot S_i and player j plays ballot B_j equals:

$$U_i(S_i, B_j) = \begin{cases} U_i(S_i \cap B_j) & \text{if } S_i \cap B_j \neq \emptyset, \\ U_i(S_i \cup B_j) & \text{if } S_i \cap B_j = \emptyset. \end{cases}$$

If $S_i \cap B_j \neq \emptyset$, it follows that $U_i(S_i \cap B_j) > U_i(X)$ since each $x \in S_i$ satisfies $u_i(x) > U_i(X)$.

If, on the contrary, $S_i \cap B_j = \emptyset$, then we let $X \setminus S_i = B_j \cup C_j$ with $\#S_i = l$,$\#B_j = m$ and $\#C_j = k - l - m$. In other words, $X = S_i \cup B_j \cup C_j$.
If $C_j = \emptyset$, then $X = S_i \cup B_j$, so that $U_i(S_i, B_j) = U_i(X)$ as wanted.
If $C_j \neq \emptyset$, then by definition, we can write that

$$U_i(S_i \cup B_j) > U_i(X) \Longleftrightarrow \frac{1}{l+m} \sum_{x \in S_i \cup B_j} u_i(x) > \frac{1}{k} \sum_{x \in X} u_i(x),$$

which is equivalent to

$$(k-l-m)(\sum_{x \in X} u_i(x)) > k \sum_{x \in C_j} u_i(x) \Longleftrightarrow \frac{1}{k} \sum_{x \in X} u_i(x) > \frac{1}{k-l-m} \sum_{x \in C_j} u_i(x).$$

The last inequality holds since, by definition, any $x \in C_j$ satisfies $u_i(x) < U_i(X)$, proving the claim. **Q.E.D.**

C Appendix: Proof of Theorem 5

We let $k = 3$ so that $X = \{a, b, c\}$. W.l.o.g. we assume that $u_1(a) > u_1(b) > u_1(c)$.

It follows that this player has the following ranking over the different winning sets. If $u_1(b) > U_1(\{a, c\})$, then:

$$U_1(\{a\}) > U_1(\{a,b\}) > U_1(\{b\}) > U_1(\{a,b,c\}) > U_1(\{a,c\}) > U_1(\{b,c\}) > U_1(\{c\}),$$

whereas if the cardinal utilities satisfy $U_1(b) < U_1(\{a, c\})$, the ranking equals:

$$U_1(\{a\}) > U_1(\{a,b\}) > U_1(\{a,c\}) > U_1(\{a,b,c\}) > U_1(\{b\}) > U_1(\{b,c\}) > U_1(\{c\}).$$

We now evaluate the set of equilibria for each possible preference ordering of the player 2, and prove that it satisfies LPE.

There are six possible permutations of the three alternatives to be analyzed.

Case 1: $u_2(a) > u_2(b) > u_2(c)$**.**

The unique equilibrium outcome is the election of a.

Case 2: $u_2(a) > u_2(c) > u_2(b)$**.**

The same argument as in Case 1 proves the claim.

Case 3: $u_2(b) > u_2(a) > u_2(c)$**.**

In this case, the unique equilibrium is $B = (a; b)$, which leads to the winning set $\{a, b\}$. Moreover, this tie is not Pareto dominated by another lottery. Indeed, $U_1(\{a, b\}) > U_1(W)$ for any $W \subseteq X \setminus \{a\}$ and $U_2(\{a, b\}) > U_2(W)$ for any $W \subseteq X \setminus \{b\}$.

Case 4: $u_2(b) > u_2(c) > u_2(a)$**.**

In this case, the equilibrium hinges on the players' preference intensities towards their middle candidate.

If $U_1(\{b\}) > U_1(\{a, c\})$ and $U_2(\{c\}) > U_2(\{a, b\})$, then the equilibrium equals $B = (ab; bc)$ with LPE winning set $\{b\}$ (since b is the most preferred outcome of player 2).

If $U_1(\{b\}) > U_1(\{a, c\})$ and $U_2(\{c\}) < U_2(\{a, b\})$, then the set of equilibria equals $B = (a; b)$ with LPE winning set $\{a, b\}$ and $B' = (ab; bc)$ with LPE winning set $\{b\}$.

If $U_1(\{b\}) < U_1(\{a,c\})$ and $U_2(\{c\}) > U_2(\{a,b\})$, then the equilibrium set equals $B = (a;bc)$ with LPE winning set $\{a,b,c\}$.
If $U_1(\{b\}) < U_1(\{a,c\})$ and $U_2(\{c\}) < U_2(\{a,b\})$, then the equilibrium equals $B = (a;b)$ with LPE winning set $\{a,b\}$.

Case 5: $u_2(c) > u_2(a) > u_2(b)$**.**

This case is symmetric to Case 4, switching the role of player 1 and 2, proving the claim.

Case 6: $u_2(c) > u_2(b) > u_2(a)$**.**

The equilibrium hinges again on the preference intensities towards their middle ranked candidate.
If $U_1(\{b\}) > U_1(\{a,c\})$ and $U_2(\{b\}) > U_2(\{a,c\})$, then the equilibrium equals $B = (ab;bc)$ with LPE winning set $\{b\}$.
If $U_1(\{b\}) > U_1(\{a,c\})$ and $U_2(\{b\}) < U_2(\{a,c\})$, then the equilibrium set equals $B = (ab;c)$ with LPE winning set $\{a,b,c\}$.
If $U_1(\{b\}) < U_1(\{a,c\})$ and $U_2(\{b\}) > U_2(\{a,c\})$, then the equilibrium equals $B = (a;bc)$ with LPE winning set $\{a,b,c\}$.
If $U_1(\{b\}) < U_1(\{a,c\})$ and $U_2(\{b\}) < U_2(\{a,c\})$, then the equilibrium equals $B = (a;c)$ with LPE winning set $\{a,c\}$.

Hence, for each possible preference ordering and preference intensities of the players towards their middle rank candidate, the winning sets at equilibrium satisfy LPE, concluding the proof.

D Appendix: Proof of the Example of Insincerity as a Strict Best Response

We now prove that the strategy profile $\sigma = (\sigma_1, \sigma_2)$ is an equilibrium. Note that since we assume that players are partially honest, no player can be indifferent between a sincere and a non-sincere strategy, since partial honesty would remove the equilibrium.

Moreover, it is weakly dominated for a player to vote for his preferred alternative and never approve his least preferred one. For the sake of presentation, only the weakly undominated strategies are considered in the tables. The table on the left side presents the expected payoffs for each pure weakly undominated strategy for player 1, whereas the second table presents those for player 2.

	e	be	abe	$U_1(\cdot,\sigma_2)$
ac	ace	$abce$	a	83.4375
acd	$acde$	$abcde$	a	83.4375
a	ae	abe	a	82.0833
ab	abe	b	ab	80
abc	$abce$	b	ab	80.3125
$abcd$	$abcde$	b	ab	80.25
ad	ade	$abcde$	a	82.5

	ac	acd	$U_2(\sigma_1,\cdot)$
e	ace	$acde$	65
be	$acde$	$abcde$	65
ae	a	a	65
ce	c	c	35
bce	c	c	35
ace	ac	ac	50
$abce$	ac	ac	50

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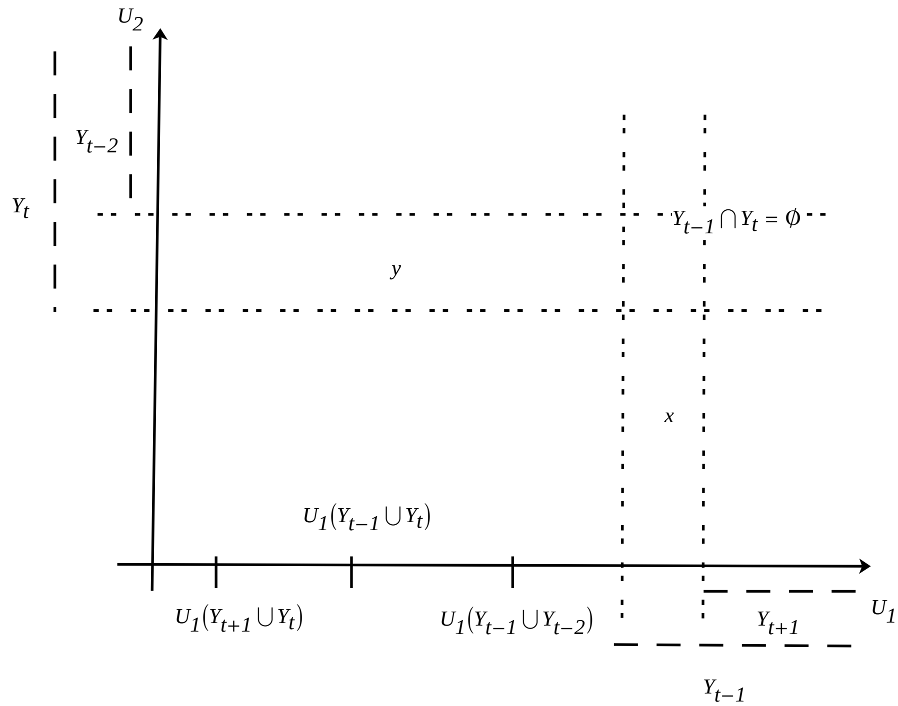

Figure 1: Existence of Equilibrium with Non-Consensual Best Responses.

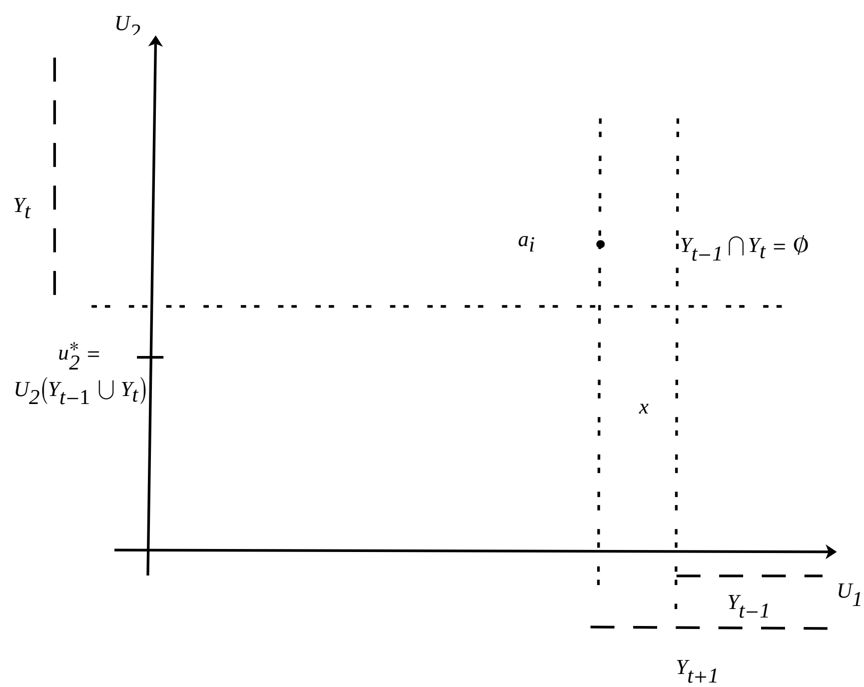

Figure 2: Existence of Equilibrium with Consensual Best Responses.